

# Standard Errors

Linear and Generalized Linear Models

# Learning Objectives

- Standard Errors
  - Linear Regression
  - GLM
- Sampling Distributions

# Linear Regression

# Standard Errors

- Find the variance of the estimate
- Find the information matrix
- Use for Inference

# Finding the Variance

$$y_i \stackrel{\text{iid}}{\sim} \mathcal{N}(\beta_0 + \beta_1 x_i, \sigma^2)$$

$$\bar{y} = \frac{1}{n} \sum y_i$$

$$\bar{y} \sim \mathcal{N}(\beta_0 + \beta_1 \bar{x}, \frac{\sigma^2}{n})$$

$$\hat{\beta}_0 = \bar{y} - \hat{\beta}_1 \bar{x}$$

$$\begin{aligned} \text{Var}(\hat{\beta}_0) &= \text{Var}(\bar{y} - \hat{\beta}_1 \bar{x}) \\ &= \text{Var}(\bar{y}) + \bar{x}^2 \text{Var}(\hat{\beta}_1) \\ &= \frac{\sigma^2}{n} + \bar{x}^2 \text{Var}(\hat{\beta}_1) \end{aligned}$$

$$\text{Var}(\hat{\beta}_1)$$

$$\hat{\beta}_1 = \frac{\sum (x_i - \bar{x})(y_i - \bar{y})}{\sum (x_i - \bar{x})^2}$$

$$\frac{1}{\left(\sum (x_i - \bar{x})^2\right)^2} \sum (x_i - \bar{x})^2 \text{Var}(y_i - \bar{y})$$

$$\text{Var}(\hat{\beta})$$

$$\text{SE}(\hat{\beta}) = \sqrt{\text{Var}(\hat{\beta})}$$

**Estimate for  $\sigma^2$**

$$Y = \beta_0 + \beta_1 X + \varepsilon$$

$$\hat{\sigma}^2 = \frac{1}{n-2} \sum_{i=1}^n (Y_i - X_i^T \hat{\beta})^2$$

$$\varepsilon \sim N(0, \sigma^2)$$

$$\hat{\sigma}^2$$

Model error

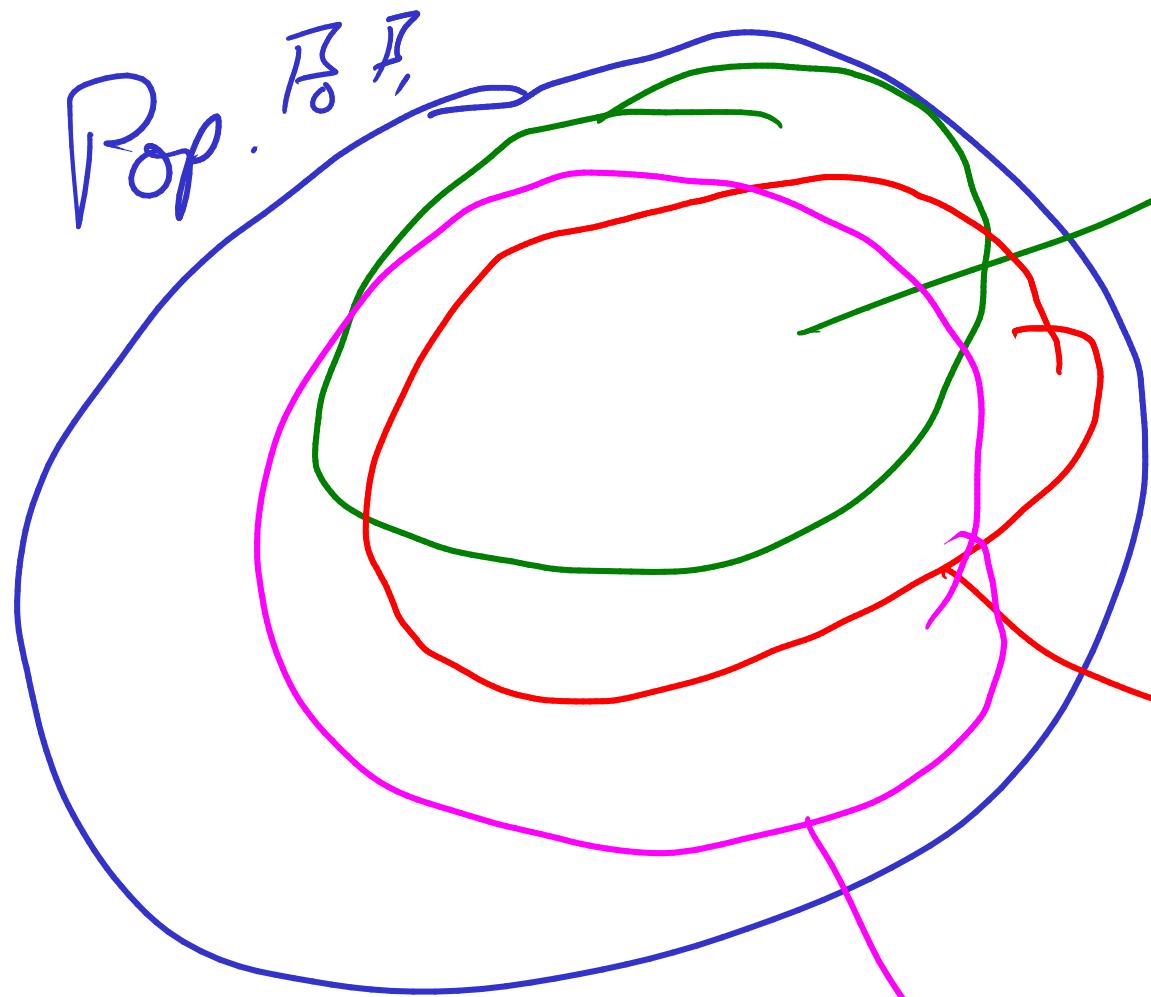


# Standard Errors of $\beta$ 's

$$SE(\hat{\beta}_0) = \sqrt{\frac{\sum_{i=1}^n x_i^2 \hat{\sigma}^2}{n \sum_{i=1}^n (x_i - \bar{x})^2}}$$

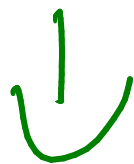
$$SE(\hat{\beta}_1) = \sqrt{\frac{\hat{\sigma}^2}{\sum_{i=1}^n (x_i - \bar{x})^2}}$$

Linear Regression.



Pop.  $\beta_0, \beta_1$

$$(x_i, y_i)_{i=1}^n$$



$$LR \rightarrow \hat{\beta}_0, \hat{\beta}_1$$

$$(x_i, y_i)$$



$$LR \rightarrow \beta_0, \beta_1$$

$$(x_i, y_i) \rightarrow LR \rightarrow \hat{\beta}_0, \hat{\beta}_1$$

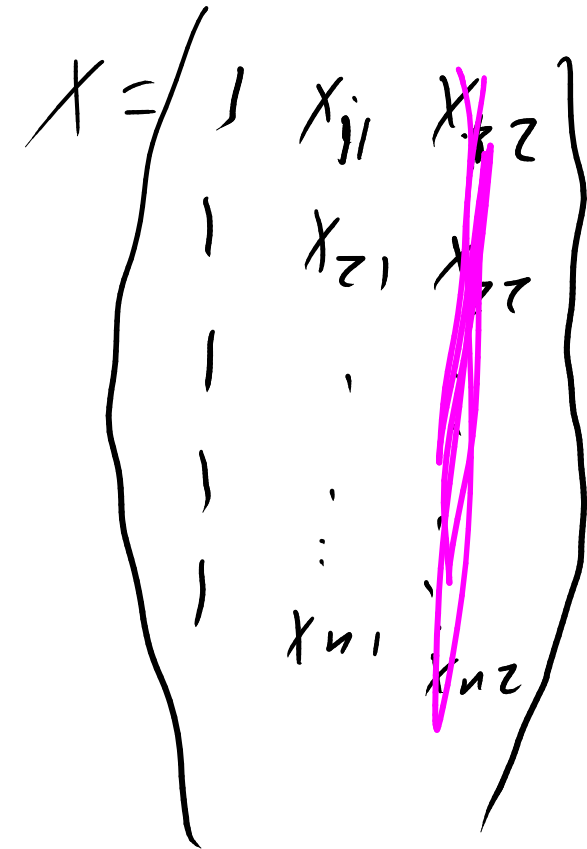
$$\hat{\beta}_1, \hat{\beta}_0, SE(\hat{\beta}_0), SE(\hat{\beta}_1)$$

# Standard Errors Matrix Form

$$\text{Var}(\hat{\beta}) = (X^T X)^{-1} \hat{\sigma}^2$$



A hand-drawn diagram showing a vertical vector of parameters enclosed in large parentheses. The elements of the vector are labeled from top to bottom as  $\beta_0$ ,  $\beta_1$ ,  $\beta_2$ , and  $\beta_n$ . An arrow points from the right side of the parentheses towards the equation above.



A hand-drawn diagram of the design matrix  $X$ , enclosed in large parentheses. The first column contains a series of ones. The second column contains elements  $x_{11}, x_{21}, \dots, x_{n1}$ . The third column contains elements  $x_{12}, x_{22}, \dots, x_{n2}$ . The second and third columns are crossed out with pink scribbles.

# Generalized Linear Models

# Large Sample Theory

Let  $X_1, \dots, X_n$  be a random sample from a distribution with parameter  $\theta$ . Let  $\hat{\theta}$  be the MLE estimator for a parameter  $\theta$ . As  $n \rightarrow \infty$ , then  $\hat{\theta}$  has a normal distribution with mean  $\theta$  and variance  $1/nI(\theta)$ , where

*[Math Processing Error]*

$$I(\beta_0) = -E \left( \frac{d^2 \ell(\beta_1)}{d\beta_1^2} \right)$$
$$\sqrt{\frac{1}{n} I(\beta_0)} = SE(\beta_0) \quad \text{GLM}$$

# Sampling Distributions

# Sampling Distributions *GLM*

$\phi$  known *B, normal/Poisson*

$\phi$  unknown *Normal/Gamma*

$$\frac{\hat{\beta}_j - \beta_j}{\text{se}(\hat{\beta}_j)} \sim N(0, 1)$$

$$\frac{\hat{\beta}_j - \beta_j}{\text{se}(\hat{\beta}_j)} \sim t_{n-p'}$$

$$\hat{\beta}_j \sim N(\beta_j, \text{SE}(\hat{\beta}_j)^2)$$

$$p' = p + 1$$

$p = \#$  predictors

$X_1, X_2, X_3, \dots$

