Maximum Likelihood Estimators

Learning Outcomes

- Maximum Likelihood Estimators
- Properties

Background Information

Estimators

An *estimator* is an operation computing the value of an estimate, that targets the parameter, using measurements from a sample.

$\underline{X} = \{x_1, \ldots, x_n\}$

$F(\theta)$

$\hat{\theta} = g(\underline{X})$

Data

known

Dirichlet
Pois
Normal

Let $X_1, \ldots, X_n \overset{iid}{\sim} F(\boldsymbol{\theta})$ where $F(\cdot)$ is a known distribution function and $\boldsymbol{\theta}$ is a vector of parameters. Let $\boldsymbol{X} = (X_1, \ldots, X_n)^{\mathrm{T}}$, be the sample collected.

$$\hat{\theta} = \underset{\theta}{\operatorname{argmax}}\, L(\theta; \bar{X})$$

$$\ell(\theta; \bar{X}) = \ln(L(\theta; \bar{X}))$$

MLE Properties

Unbiased Estimators

Let $X_1, \ldots, X_n$ be a random (iid) sample from a distribution with parameter θ. Let $\hat{\theta}$ be an estimator for a parameter θ. Then $\hat{\theta}$ is an unbiased estimator if $E(\hat{\theta}) = \theta$. Otherwise, $\hat{\theta}$ is considered biased.

$$\hat{\theta} = g(\bar{X})$$

$$E(\hat{\theta}) = E(g(\bar{X})) = \theta$$

Consistent Estimators

Let $X_1, \ldots, X_n$ be a random sample from a distribution with parameter θ. The estimator $\hat{\theta}$ is a consistent estimator of the θ if

1. $E\{(\hat{\theta} - \theta)^2\} \rightarrow 0$ as $n \rightarrow \infty$
2. $P(|\hat{\theta} - \theta| \geq \epsilon) \rightarrow 0$ as $n \rightarrow \infty$ for every $\epsilon > 0$

Invariance Property

If $\hat{\theta}$ is an ML estimator of θ, then for any one-to-one function g, the ML estimator for $g(\theta)$ is $g(\hat{\theta})$.

Large Sample Theory

Let $X_1, \ldots, X_n$ be a random sample from a distribution with parameter θ. Let $\hat{\theta}$ be the MLE estimator for a parameter θ. As $n \to \infty$, then $\hat{\theta}$ has a normal distribution with mean θ and variance $1/nI(\theta)$, where

$$I(\theta) = E\left[-\frac{\partial^2}{\partial\theta^2}\log\{f(X;\theta)\}\right]$$

Example

Exponential Distribution

Let $X_1, \ldots, X_n \overset{iid}{\sim} Exp(\lambda)$. Find the sampling distribution of the MLE of λ

$$f(x) = \lambda e^{-x\lambda}$$

$$\ln(f(x)) = \ln(\lambda) - x\lambda$$

$$= \frac{1}{\lambda} - x$$

$$= -\frac{1}{\lambda^2}$$

$$E\left(-\,-\frac{1}{\lambda^2}\right)$$

$$E\left(\frac{1}{\lambda^2}\right) = \frac{1}{\lambda^2}$$

$$\hat{\lambda} \sim N\left(\lambda, \frac{1}{nI(\lambda)}\right)$$

$$I(\lambda) = \frac{1}{\lambda^2}$$

$$\hat{\lambda} \sim N\left(\lambda, \frac{\lambda^2}{n}\right)$$

$\hat{\lambda}$ $\quad X_1, \ldots, X_n \sim Exp(\lambda)$ $\quad f(x) = \lambda e^{-\lambda x}$

1. Find $L(\lambda)$
2. Find $\ell(\lambda)$
3. Derivative
4. set to 0

$$L(\lambda) = \prod_{i=1}^{n} \lambda e^{-\lambda x_i}$$

$$\ell(\lambda) = \ln(L(\lambda)) = \sum_{i=1}^{n} \ln\left(\lambda e^{-\lambda x_i}\right)$$

$$= \sum_{i=1}^{n} \ln(\lambda) + \ln\left(e^{-\lambda x_i}\right)$$

$$= \sum_{i=1}^{n} \ln(\lambda) - \lambda x_i$$

$$\ell(\lambda) = n \ln(\lambda) - \lambda \sum_{i=1}^{n} x_i$$

$$\frac{d\,\ell(\lambda)}{d\lambda} = \frac{n}{\lambda} - \sum_{i=1}^{n} x_i = 0$$

$$\frac{n}{\lambda} = \sum_{i=1}^{n} x_i$$

$$\hat{\lambda} = \frac{n}{\sum_{i=1}^{n} x_i}$$

$$f(x;\beta) = \frac{1}{\beta} e^{-x\frac{1}{\beta}}$$

Poisson Distribution

Let $X_1, \ldots, X_n \overset{iid}{\sim} \text{Pois}(\lambda)$, Find the sampling distribution of the MLE.

$\hat{\lambda} = \bar{X}$

$f(x) = \dfrac{e^{-\lambda} \lambda^{x}}{x!}$

$\bar{X} \sim N\left(\lambda, \dfrac{\lambda}{n}\right)$

$\ln(f(x)) = \ln\left(\dfrac{e^{-\lambda} \lambda^{x}}{x!}\right)$

$I(\lambda) = \dfrac{1}{\lambda},$

$= -\lambda + x\ln(\lambda) - \ln(x!)$

$= -1 + \dfrac{x}{\lambda} - 0$

$Var(\hat{\theta}) = \dfrac{1}{nI(\theta)}$

$= -\dfrac{x}{\lambda^2}$

$$E\left(\frac{X}{\lambda^2}\right) = \frac{1}{\lambda^2} E(X)$$

$$= \frac{1}{\lambda^2} \lambda$$

$$I(\lambda) = \frac{1}{\lambda}$$

Normal Distribution

$\bar{X} \sim N\left(\mu, \sigma^2/n\right)$

$Var(X) = E(X^2) - E(X)^2$ $\quad$ $E(X^2) = Var(X) + E(X)^2$

Let $X_1, \ldots, X_n \overset{iid}{\sim} N(\mu, \sigma^2)$. Are the MLE's of μ and σ^2 unbiased?

$\hat{\mu} = \bar{X}$ $\quad$ $\hat{\sigma}^2 = \frac{1}{n}\sum (x_i - \bar{x})^2$

$$E(\hat{\sigma}^2) = E\left(\frac{1}{n}\sum (x_i - \bar{x})^2\right) = \frac{1}{n}E\left(\sum (x_i - \bar{x})^2\right)$$

$$\frac{1}{n}E\left(\sum_{i=1}^{n}\left(x_i^2 - 2x_i\bar{x} + \bar{x}^2\right)\right) = \frac{1}{n}E\left(\sum x_i^2 - \sum 2x_i\bar{x} + n\bar{x}^2\right)$$

$$\frac{1}{n}E\left(\sum x_i^2 - 2\bar{x}\sum x_i + n\bar{x}^2\right) = \frac{1}{n}E\left(\sum x_i^2 \overbrace{-2n\bar{x}^2 + n\bar{x}^2}^{-n\bar{x}^2}\right)$$

$$\bar{x} = \frac{\sum x_i}{n} \Rightarrow n\bar{x} = \sum x_i$$

$$\frac{1}{n}\left[E\left(\sum x_i^2\right) - nE(\bar{X}^2)\right] \Rightarrow \frac{1}{n}\left[\sum E(x_i^2) - nE(\bar{x}^2)\right]$$

$$\Rightarrow \frac{1}{n}\left[\sum(\sigma^2+\mu^2) - n\left(\frac{\sigma^2}{n}+\mu^2\right)\right] \Rightarrow \frac{1}{n}\left[n\sigma^2 + \cancel{n\mu^2} - \sigma^2 - \cancel{n\mu^2}\right]$$

$$\Rightarrow \frac{1}{n}\left[n\sigma^2 - \sigma^2\right] = \sigma^2 - \frac{\sigma^2}{n} \Rightarrow \sigma^2\left(1-\frac{1}{n}\right) = \sigma^2\left(\frac{n-1}{n}\right)$$

$$\frac{n}{n-1}\hat{\sigma}^2 = E\left(\frac{n}{n-1}\hat{\sigma}^2\right) = \frac{n}{n-1}E(\hat{\sigma}^2)$$

$$\frac{\cancel{n}}{\cancel{n-1}}\sigma^2\frac{\cancel{n-1}}{\cancel{n}}$$

$$\frac{\cancel{n}}{n-1}\,\frac{\sum(x_i-\bar{x})^2}{\cancel{n}} \Rightarrow \frac{\sum(x_i-\bar{x})^2}{n-1} = S^2$$

$$E(\bar{X}) = E\left(\frac{1}{n}\sum X_i\right) = \frac{1}{n}E\left(\sum X_i\right) = \frac{1}{n}\sum E(X_i)$$

$$\frac{1}{n}\sum \mu = \frac{n\mu}{n} = \mu$$