Functions of Random Variables

Learning Outcomes

- Functions of Random Variables
- Finding PDFs using the distribution function
- Finding the PDF of a function of random variables
- Using Moment Generating Functions

Function of Random Variables

Function of Random Variables

$X \sim F(\theta)$ $\quad$ $Y \sim G(\theta^*)$

$\begin{pmatrix} X \\ Y \end{pmatrix} \sim F(\theta)$

$Z = X + Y$

$\uparrow$

RV

$Z \sim H(\theta^+)$

$$\bar{X} = \frac{1}{n}\sum_{i=1}^{n} X_i \qquad \bar{Y} = \frac{1}{n}\sum_{i=1}^{n} Y_i$$

$$X_i \sim F(\theta) \qquad Y_i \sim F(\theta^*)$$

$$\bar{X} \sim G(\theta^*) \qquad \bar{Y} \sim G(\theta^*)$$

$$Z = \overset{0.5}{\bar{X}} - \overset{0.5}{\bar{Y}} \qquad P(|Z| = 0) = 0.82$$

$$Z \sim H(\theta) \qquad P(|Z|) = 0.82$$

Obtaining the PDFs

Using the Distribution Function

Let there be a random variable X with a known distribution function $F_X(x)$, the density function for the random variable $Y = g(X)$ can be found with the following steps

inverse

1. Find the region of Y in the space of X, find $g^{-1}(y)$
2. Find the region of $Y \leq y$
3. Find $F_Y(y) = P(Y \leq y)$ using the probability density function of X over region $Y \leq y$
4. Find $f_Y(y)$ by differentiating $F_Y(y)$

Example 1

Let X have the following probability density function:

$$f_X(x) = \begin{cases} 2x & 0 \leq x \leq 1 \\ 0 & \text{otherwise} \end{cases}$$

Find the probability density function of $Y = 3X - 1$?

① $-1 \leq Y \leq 2$

② $Y = 3X - 1$

$X = \frac{Y+1}{3}$

③ $P(Y \le y) = P(3X - 1 \le y)$

$$P\left(X \le \frac{y+1}{3}\right) = \int_0^{\frac{y+1}{3}} 2x \, dx$$

$$x^2 \Big|_0^{\frac{y+1}{3}} = \frac{(y+1)^2}{9} = F(y)$$

$$f(y) = \frac{2(y+1)}{9} \qquad -1 \le y \le 2$$

Using the PDF

Let there be a random variable X with a known distribution function $F_X(x)$, if the random variable $Y = g(X)$ is either increasing or decreasing, than the probability density function can be found as

Jacobian

$$f_Y(y) = f_X\{g^{-1}(y)\} \left| \frac{dg^{-1}(y)}{dy} \right|$$

Example 2

Let X have the following probability density function:

$$f_X(x) = \begin{cases} \frac{3}{2}x^2 + x & 0 \leq y \leq 1 \\ 0 & \text{otherwise} \end{cases}$$

Find the probability density function of $Y = 5 - (X/2)$?

① $X = (Y-5)/(-2) = -2Y + 10 \leftarrow g^{-1}(y)$

② -2 $\quad f_X(g^{-1}(y)) \left| \frac{d g^{-1}(y)}{dy} \right|$

③ $f_Y = \left[\frac{3}{2}(-2Y+10)^2 - 2Y + 10\right] \cdot |-2|$

$$f(y) = 3(-2y+10)^2 - 4y + 20$$

MGF Properties: Linearity

Let X follow a distribution f, with ~~the~~ an MGF $M_X(t)$, the MGF of $Y = aX + b$ is given as

$$M_Y(t) = e^{tb} M_X(at)$$

MGF Properties: Uniqueness

Let X and Y have the following distributions $F_X(x)$ and $F_Y(y)$ and MGFs $M_X(t)$ and $M_Y(t)$, respectively. X and Y have the same distribution $F_X(x) = F_Y(y)$ if and only if $M_X(t) = M_Y(t)$.

$$M_X = M_Y \iff F_X = F_Y$$

X Y are the same RV

Using the MGF

Using the uniqueness property of Moment Generating Functions, for a random variable X with a known distribution function $F_X(x)$ and random variable $Y = g(X)$, the distribution of Y can be found by:

1. Find the moment generating function of Y, $M_Y(t)$.
2. Compare $M_Y(t)$, with known moment generating functions. If $M_Y(t) = M_V(t)$, for all values t, the~~m~~n Y and V have identical distributions.

Example 3

Let X follow a normal distribution with mean μ and variance σ^2. Find the distribution of $Z = \underbrace{\frac{X-\mu}{\sigma}}$.

$Z = \frac{1}{\sigma} X - \mu/\sigma$

$M_Z(t) = e^{-\frac{\mu}{\sigma}t} M_X\left(\frac{1}{\sigma}t\right)$

$e^{-\cancel{\frac{\mu}{\sigma}t}} \quad e^{\cancel{\mu\frac{1}{\sigma}t} + \cancel{\sigma^2}\left(\frac{t^2}{\cancel{\sigma^2}}\right)/2}$

$\underbrace{M_X(t)} = \underbrace{e^{\mu t + \frac{\sigma^2 t^2}{2}}}$

$\mu = 0 \quad \sigma^2 = 1$

$e^{0t + \frac{1 t^2}{2}}$

$M_Z(t) = \underbrace{e^{t^2/2}} \longleftrightarrow e^{t^2/2}$

$Z \sim N(0,1)$ ← Standard Normal Distribution

Example 4

Let Z follow a standard normal distribution with mean 0 and variance 1. Find the distribution of $Y = Z^2$

$Z \sim N(0,1)$ $\qquad Y \sim ?$

$E(e^{tY}) = E(e^{tZ^2}) = \int_{-\infty}^{\infty} e^{tz^2} \frac{1}{\sqrt{2\pi}} e^{-z^2/2}\, dz$

$\frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} e^{tz^2 - z^2/2}\, dz \Rightarrow \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} e^{-\frac{z^2}{2}(2t+1)}\, dz$

$\mu =$

$\cancel{\frac{1}{\sqrt{2\pi}}} \left[\cancel{\sqrt{2\pi}} \sqrt{(2t+1)^{-1}} \right]$

$\int_{-\infty}^{\infty} e^{-\frac{x^2}{2\sigma^2}}\, dx = \sqrt{2\pi\sigma^2}$

$$M_Y(t) = (2t+1)^{-1/2}$$

$$\frac{1}{\sigma^2} = (2t+1)$$

$$\sigma^2 = (2t+1)^{-1}$$

$$\chi^2(k) \quad M(t) = (2t+1)^{-k/2}$$

$$\sqrt{2\pi(2t+1)^{-1}}$$

$$Y \sim \chi^2(1)$$